DISLOCATED TOPOLOGIES[1]

Pascal Hitzler — Anthony Karel Seda *

We study a generalized notion of topology which evolved from applications in the area of logic programming semantics. The generalization is obtained by relaxing the requirement that a neighbourhood of a point includes the point itself, and by allowing neighbourhoods of points to be empty. We show that it is meaningful to discuss neighbourhoods, convergence, and continuity in these spaces. A generalized version of the Banach contraction mapping theorem can also be established. We will also show how the generalized metrics studied here can be obtained from conventional metrics.

K e y w o r d s: Generalized topology, generalized metric, generalized Banach contraction mapping theorem

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1 INTRODUCTION

In recent years, the role of topology in Logic Programming has come to be recognized (see e.g. [1,2,6]). In particular, topological methods are employed in order to obtain fixed-point semantics for logic programs. The *dislocated metric spaces* which we wish to present in this paper are motivated by such considerations.

The plan of the paper is as follows. In Section 2, we will first define *dislocated metrics* and state a generalization of the Banach contraction mapping theorem which has found applications in the area of programming language semantics. In Section 3, we will study *dislocated topologies* which generalize conventional topologies and can be understood as underlying the notion of dislocated metric. In Section 4, we will further investigate dislocated metrics and their relationships with the concepts introduced in Section 3 and with conventional metrics. Finally, in Section 5, we will conclude with a short discussion.

2 A GENERALIZED BANACH FIXED–POINT THEOREM

We will define dislocated metrics and present a generalization of the Banach contraction mapping theorem, Theorem 2.2, for these spaces. This theorem has been applied in the area of logic programming semantics, and the interested reader can find details of this in the long version of the paper[1].

Let X be a set and let $\varrho\colon X \times X \to \mathbb{R}_0^+$ be a function, called a *distance function*. Consider the following conditions:

(Mi) For all $x \in X$, $\varrho(x,x) = 0$.
(Mii) For all $x,y \in X$, if $\varrho(x,y) = 0$, then $x = y$.
(Miii) For all $x,y \in X$, $\varrho(x,y) = \varrho(y,x)$.
(Miv) For all $x,y,z \in X$, $\varrho(x,y) \le \varrho(x,z) + \varrho(z,y)$.
(Miv$'$) For all $x,y,z \in X$, $\varrho(x,y) \le \max\{\varrho(x,z), \varrho(z,y)\}$.

If ϱ satisfies conditions (Mi) to (Miv), then it is called a *metric*. If it satisfies conditions (Mi), (Miii) and (Miv), it is called a *pseudo-metric*. If it satisfies (Mii), (Miii) and (Miv), we will call it a *dislocated metric* (or simply *d-metric*). If a (pseudo-, d-) metric satisfies the strong triangle inequality (Miv$'$), then it is called a (pseudo-, d-) *ultra*metric.

Dislocated metrics were studied under the name of *metric domains* in the context of domain theory in [3], where proofs of the results of this section can be found. The slightly less general notion of *partial metric* was also studied in [4]. We proceed now with the definitions needed for stating the generalized Banach contraction mapping theorem. As it turns out, these notions can be carried over directly from conventional metrics.

A sequence (x_n) in a d-metric ϱ *converges with respect to* ϱ (or *in* ϱ) if there exists an $x \in X$ such that $\varrho(x_n,x)$ converges to 0 as $n \to \infty$. In this case, x is called the *limit* of (x_n) (*in* ϱ) and we write $x_n \to x$. A sequence (x_n) in a d-metric space is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m,n \ge n_0$ we have $\varrho(x_m,x_n) < \varepsilon$. A d-metric space (X,ϱ) is called *complete* if every Cauchy sequence in X converges with respect to ϱ. A function $f\colon X \to X$ is called a *contraction* if there exists $0 \le \lambda < 1$ such that $\varrho(f(x),f(y)) \le \lambda\varrho(x,y)$ for all $x,y \in X$.

2.1 Proposition. *The following statements hold.*
(a) *Limits in d-metric spaces are unique.*

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[1] A longer version of this paper is available from Web: http://maths.ucc.ie/~ {pascal,seda} /index.html

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(b) *Every converging sequence in a d-metric space is a Cauchy sequence.*

2.2 Theorem. *Let (X,ϱ) be a complete d-metric space and let $f\colon X \to X$ be a contraction. Then f has a unique fixed point.*

A proof of this theorem was given in [3]. We will give an alternative proof in Section 4 which is more in the spirit of the proof of the original Banach contraction mapping theorem.

3 DISLOCATED TOPOLOGIES

We are interested in investigating dislocated metrics from a topological point of view following the inspiration provided by Section 2. Since constant sequences do not in general converge in d-metric spaces, a conventional topological approach is not feasible, and the conventional notions of neighbourhood, convergence and continuity need to be modified.

3.1 Definition. An (*open ε-*)*ball* in a d-metric space (X,ϱ) with *centre* $x \in X$ is a set of the form $B_\varepsilon(x) = \{y \in X \mid \varrho(x,y) < \varepsilon\}$, where $\varepsilon > 0$.

Note that a ball may be empty in a d-metric space. In fact, the above definition of ball does not imply that the centre of a ball is contained in the ball itself: the point may be *dislocated* from the ball, and hence our usage of the term "dislocated" in this paper.

3.2 Proposition. *Let (X,ϱ) be a d-metric space.*

(a) *The following three conditions are equivalent:*

(i) *For all $x \in X$, we have $\varrho(x,x) = 0$.*

(ii) *ϱ is a metric.*

(iii) *For all $x \in X$ and all $\varepsilon > 0$, we have $B_\varepsilon(x) \neq \emptyset$.*

(b) *The space (X',ϱ), where $X' = \{x \in X \mid \varrho(x,x) = 0\}$, is a metric space.*

P r o o f . (a) That (i) implies (ii) is obvious, as is (ii) implies (iii). We show (iii) implies (i). Since $B_\varepsilon(x) \neq \emptyset$ for all $\varepsilon > 0$, there exists, for each $\varepsilon > 0$, some $y \in X$ with $\varrho(x,y) < \varepsilon$. But, for all $y \in X$, we have $\varrho(x,x) \leq 2 \cdot \varrho(x,y)$, and hence $\varrho(x,x) < \varepsilon$ for all $\varepsilon > 0$. Therefore, $\varrho(x,x) = 0$.

(b) Obviously, (X',ϱ) is a d-metric space. The assertion now follows immediately from (a).

We proceed next with the investigation of dislocated metrics from the topological point of view. If X is a set, then a relation $\lessdot\ \subseteq X \times \mathcal{P}(X)$ (written infix) is called a *d-membership relation* (*on* X) if it satisfies the following property for all $x \in X$ and $A, B \subseteq X$: *whenever $x \lessdot A$ and $A \subseteq B$, we have $x \lessdot B$*. We say x *is below* A if $x \lessdot A$. The "below"-relation is a generalization of the membership relation from set theory, which will allow us to define a suitable notion of neighbourhood.

3.3 Definition. Let X be a set, let $\lessdot$ be a d-membership relation on X and let $\mathcal{U}_x \neq \emptyset$ be a collection of subsets of X for each $x \in X$. We call $(\mathcal{U}_x, \lessdot)$ a *d-neighbourhood system* (*d-nbhood system*) *for* x if it satisfies the following conditions.

(Ni) If $U \in \mathcal{U}_x$, then $x \lessdot U$.

(Nii) If $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.

(Niii) If $U \in \mathcal{U}_x$, then there is a $V \subseteq U$ with $V \in \mathcal{U}_x$ such that for all $y \lessdot V$ we have $U \in \mathcal{U}_y$.

(Niv) If $U \in \mathcal{U}_x$ and $U \subseteq V$, then $V \in \mathcal{U}_x$.

Each $U \in \mathcal{U}_x$ is called a *d-neighbourhood* (*d-nbhood*) of x. Finally, let X be a set, let $\lessdot$ be a d-membership relation on X and for each $x \in X$ let $(\mathcal{U}_x, \lessdot)$ be a d-nbhood system for x. Then $(X, \mathcal{U}, \lessdot)$ (or simply X) is called a *d-topological space*, where $\mathcal{U} = \{\mathcal{U}_x \mid x \in X\}$.

Note that points may have empty d-nbhoods and also that Definition 3.3 is exactly the definition of a topological neighbourhood system if $\lessdot$ is the membership relation $\in$. Proposition 3.4, next, shows that d-nbhood systems arise naturally from d-metrics.

3.4 Proposition. *Let (X,ϱ) be a d-metric space. Define the d-membership relation $\lessdot$ as the relation $\{(x,A) \mid$ there exists $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq A\}$. For each $x \in X$, let $\mathcal{U}_x$ be the collection of all subsets A of X such that $x \lessdot A$. Then $(\mathcal{U}_x, \lessdot)$ is a d-nbhood system for x.*

P r o o f . It is easy to see that $\lessdot$ is indeed a d-membership relation.

(Ni) is obvious. Note that we also have the reverse property: if $x \lessdot U$, then $U \in \mathcal{U}_x$.

(Nii) If $x \lessdot U, V$, then there are balls A, B with centre x such that $A \subseteq U$ and $B \subseteq V$. Without loss of generality let A be the smaller of the balls A and B. Then $A = A \cap B \subseteq U \cap V$.

(Niii) Let $U \in \mathcal{U}_x$, that is, $x \lessdot U$. Then there is a ball B with centre x such that $B \subseteq U$ and $B \in \mathcal{U}_x$. Now let $y \lessdot B$ be arbitrary. We have to show that $y \lessdot U$. But $y \lessdot B$ implies that there is a ball B' with centre y such that $y \lessdot B' \subseteq B \subseteq U$. So $y \lessdot U$.

(Niv) This is obvious since $x \lessdot U \subseteq V$ implies $x \lessdot V$.

We note that if (X,ϱ) is a metric space, then the construction above yields the usual topology associated with a metric.

Once the notion of d-nbhood is defined, it is straightforward to adapt the notion of convergence to d-topological spaces, as follows.

3.5 Definition. Let $(X, \mathcal{U}, \lessdot)$ be a d-topological space and let $x \in X$. A (topological) net (x_λ) *d-converges to* $x \in X$ if, for each d-nbhood U of x, we have that x_λ is *eventually in* U, that is, there exists λ_0 such that $x_\lambda \in U$ for all $\lambda > \lambda_0$.

Note that if for some $x \in X$ we have $\emptyset \in \mathcal{U}_x$, then the constant sequence (x) does not d-converge. In fact, if $\emptyset \in \mathcal{U}_x$, then no net in X d-converges to x. Note also that the notion of convergence obtained in Definition 3.5 is a natural generalization of convergence with respect to a d-metric, and we investigate this next.

3.6 Proposition. *Let (X,ϱ) be a d-metric space and let $(X,\mathcal{U},\lessdot)$ be the d-topological space obtained from it via the construction given in Proposition* 3.4. *Let (x_n) be a sequence in X. Then (x_n) converges in ϱ if and only if (x_n) d-converges in $(X,\mathcal{U},\lessdot)$.*

Proof. Let (x_n) be convergent in ϱ to some $x \in X$, so that $\varrho(x_n,x) \to 0$ as $n \to \infty$, and let U be a d-nbhood of x. Then there exists $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq U$. Since $\varrho(x_n,x) \to 0$, there exists n_0 such that $x_n \in B \subseteq U$ for all $n > n_0$ and hence (x_n) d-converges to x.

Conversely, let (x_n) be d-convergent to some $x \in X$. Thus, for each d-nbhood U of x there exists n_0 such that $x_n \in U$ for each $n > n_0$. For each $\varepsilon > 0$, $B_\varepsilon(x)$ is a d-nbhood of x. Since ε can be chosen arbitrarily small, we must have $\varrho(x_n,x) \to 0$ for $n \to \infty$.

We proceed with defining continuity on d-topological spaces.

3.7 Definition. Let X and Y be d-topological spaces and let $f\colon X \to Y$ be a function. Then f is *d-continuous at* $x_0 \in X$ if, for each d-nbhood V of $f(x_0)$ in Y, there is a d-nbhood U of x_0 in X such that $f(U) \subseteq V$. We say that f is *d-continuous on* X if f is d-continuous at each $x_0 \in X$.

The following theorem shows that the notion of d-convergence can be characterized in terms of nets, by analogy with conventional topology.

3.8 Theorem. *Let X and Y be d-topological spaces and let $f\colon X \to Y$ be a function. Then f is d-continuous if and only if for each net (x_λ) in X which d-converges to some $x_0 \in X$, $(f(x_\lambda))$ is a net in Y which d-converges to $f(x_0) \in Y$.*

Proof. Let f be d-continuous at x_0 and let x_λ be a net which d-converges to x_0. Let V be a d-nbhood of $f(x_0)$. Then there exists a d-nbhood U of x_0 such that $f(U) \subseteq V$. Since x_λ is eventually in U, we obtain that $f(x_\lambda)$ is eventually in V, and hence $f(x_\lambda)$ d-converges to $f(x_0)$.

Conversely, if f is not d-continuous at x_0, then for some d-nbhood V of $f(x_0)$ and for all $U \in \mathcal{U}_{x_0}$ we have $f(U) \nsubseteq V$. Thus for each $U \in \mathcal{U}_{x_0}$ there is an $x_U \in U$ with $f(x_U) \notin V$. Then (x_U) is a net in X which d-converges to x_0 whilst $f(x_U)$ does not d-converge to $f(x_0)$.

4 DISLOCATED METRICS

In Section 3, we have generalized convergence from d-metrics to d-topologies. However, we still lack a notion of continuity for d-metrics. We will investigate this next, and this will enable us to give a proof of Theorem 2.2 which is analogous to the standard proof of the Banach contraction mapping theorem.

4.1 Proposition. *Let (X,ϱ) and (Y,ϱ') be d-metric spaces, let $f\colon X \to Y$ be a function and let $(X,\mathcal{U},\lessdot)$ and $(Y,\mathcal{V},\lessdot')$ be the d-topological spaces obtained from (X,ϱ), respectively (Y,ϱ'), via the construction in Proposition* 3.4. *Then f is d-continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.*

Proof. Let f be d-continuous at $x_0 \in X$ and let $\varepsilon > 0$. Then $B_\varepsilon(f(x_0))$ is a d-nbhood of $f(x_0)$. By definition of d-continuity, there exists a d-nbhood U of x_0 with $f(U) \subseteq B_\varepsilon(f(x_0))$. But since U is a d-nbhood of x_0, there exists a ball $B_\delta(x_0) \subseteq U$ and therefore $f(B_\delta(x_0)) \subseteq f(U) \subseteq B_\varepsilon(f(x_0))$.

Conversely, assume that the ε-δ-condition on f holds and let V be a d-nbhood of $f(x_0)$. Then there exists $\varepsilon > 0$ with $B_\varepsilon(f(x_0)) \subseteq V$ and $\delta > 0$ with $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0)) \subseteq V$. Since $B_\delta(x_0)$ is a d-nbhood of x_0 we obtain d-continuity of f.

4.2 Proposition. *Let (X,ϱ) be a d-metric space, let $f\colon X \to X$ be a contraction and let $(X,\mathcal{U},\lessdot)$ be the d-topological space obtained from (X,ϱ) via the construction in the proof of Proposition* 3.4. *Then f is d-continuous.*

Proof. Let $x_0 \in X$ and let $\varepsilon > 0$ be arbitrarily chosen. For $\delta = \frac{\varepsilon}{\lambda+1}$, we obtain $d(f(x),f(x_0)) \le \lambda d(x,x_0) \le \lambda\frac{\varepsilon}{\lambda+1} < \varepsilon$ for all $x \in B_\delta(x_0)$, and therefore $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$ as required.

Proof of Theorem 2.2. With our preparations, the proof follows the proof of the Banach contraction mapping theorem on metric spaces, and we only sketch the details here.

Let $x \in X$ be arbitrarily chosen. Then the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and converges in (X,ϱ) to some point y. Since f is a contraction, it is also d-continuous by Proposition 4.2 from which we obtain $y = \lim f^n(x) = f(\lim f^{n-1}(x)) = f(y)$ by Theorem 3.8. Uniqueness follows since if z is a fixed point of f, then $\varrho(x,z) = \varrho(f(x),f(z)) \le \lambda\varrho(x,z)$ and therefore $\varrho(x,z) = 0$, and hence $x = z$ by (Mii).

It is a corollary of the proof just given that iterates of any point converge to the unique fixed point of the function in question. In denotational semantics, this additional feature of the fixed-point theorem is desirable since it yields a method of actually obtaining the fixed point whose existence has been shown.

In the remainder of this section, we will investigate relationships between conventional metrics and d-metrics. First note that if f is a contraction on a d-metric space X, we have $\varrho(f(x),f(x)) \le \lambda\varrho(x,x)$ for all $x \in X$. Since the requirement $\varrho(x,x) = 0$ for all $x \in X$ renders a d-metric a metric, we are interested in studying the function $u_\varrho\colon X \to \mathbb{R}$ defined by $u_\varrho(x) = \varrho(x,x)$. We will call this function the *dislocation function* of ϱ.

4.3 Lemma. *Let (X, ϱ) be a d-metric space. Then $u_\varrho \colon X \to \mathbb{R}$ is d-continuous.*

Proof. Recalling the observations following Definition 3.5, let $x \in X$ and let (x_λ) be a net in X which d-converges to x, so that for each $\varepsilon > 0$ there exist λ_0 such that $\varrho(x_\lambda, x) < \varepsilon$ for all $\lambda > \lambda_0$. Since $u_\varrho(x_\lambda) = \varrho(x_\lambda, x_\lambda) \leq 2\varrho(x_\lambda, x)$ for all λ, we obtain $u_\varrho(x_\lambda) \to 0$ for increasing λ. It remains to show that $u_\varrho(x) = 0$, and this follows from $u_\varrho(x) = \varrho(x,x) \leq 2\varrho(x_\lambda, x)$, since the latter term tends to 0 for increasing λ.

The following is a general result which shows how d-metrics can be obtained from conventional metrics.

4.4 Proposition. *Let (X, d) be a metric space, let $u \colon X \to \mathbb{R}_0^+$ be a function and let $T \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be a symmetric operator which satisfies the triangle inequality. Then (X, ϱ) with*

$$\varrho(x,y) := d(x,y) + T(u(x), u(y))$$

is a d-metric space and $u_\varrho(x) = T(u(x), u(x))$ for all $x \in X$. In particular, if $T(x,x) = x$ for all $x \in \mathbb{R}$, then $u_\varrho \equiv u$.

Proof.
(Mii) If $\varrho(x,y) = 0$, then $d(x,y) + T(u(x), u(y)) = 0$. Hence $d(x,y) = 0$ and $x = y$.
(Miii) Obvious by symmetry of d and T.
(Miv) Obvious since d and T satisfy the triangle inequality.

Completeness also carries over if some continuity conditions are imposed.

4.5 Proposition. *Using the notation of Proposition 4.4, let u be continuous as a function from (X,d) to $\mathbb{R}_0^+$ (endowed with the usual topology), and let T be continuous as a function from the topological product space $\mathbb{R}^2$ to $\mathbb{R}_0^+$, satisfying the additional property $T(x,x) = x$ for all x. If (X,d) is a complete metric space, then (X, ϱ) is a complete d-metric space.*

Proof. Let (x_n) be a Cauchy sequence in (X, ϱ). Thus, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $d(x_m, x_n) \leq d(x_m, x_n) + T(u(x_m), u(x_n)) = \varrho(x_m, x_n) < \varepsilon$. So (x_n) is also a Cauchy sequence in (X, d) and therefore has a unique limit x in (X, d). In particular, we have $x_n \to x$ in (X, d) and also $u(x_n) \to u(x)$ and $T(u(x_n), u(x)) \to T(u(x), u(x)) = u(x)$. We have to show that $\varrho(x_n, x)$ converges to 0 as $n \to \infty$. For all $n \in \mathbb{N}$ we obtain $\varrho(x_n, x) = d(x_n, x) + T(u(x_n), u(x)) \to u(x) = u_\varrho(x)$, and it remains to show that $\varrho(x,x) = 0$. But this follows from the fact that (x_n) is a Cauchy sequence, since the latter implies that $u(x_n) = u_\varrho(x_n) = \varrho(x_n, x_n) \to 0$ as $n \to \infty$, and hence by continuity of u we obtain $u(x) = 0$.

We can also obtain a partial converse of Proposition 4.4.

4.6 Proposition. *Let (X, ϱ) be a d-metric space which satisfies the partial metric triangle inequality* [4] *$\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z) - \varrho(y,y)$ for all $x, y, z \in X$, and let $T \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be a symmetric operator such that $T(x,x) = x$, for all $x \in \mathbb{R}$, which satisfies the inequality*

$$T(x,y) \geq T(x,z) + T(z,y) - T(z,z)$$

for all $x, y, z \in \mathbb{R}$. Then (X, d) with

$$d(x,y) := \varrho(x,y) - T\left(u_\varrho(x), u_\varrho(y)\right)$$

is a pseudo-metric space.

Proof. (Mi) For all $x \in X$ we have $d(x,x) = \varrho(x,x) - u_\varrho(x) = 0$.
(Miii) Obvious by symmetry of ϱ and T.
(Miv) For all $x, y \in X$ we obtain

$$\begin{aligned}
&d(x,y) = \varrho(x,y) - T(u_\varrho(x), u_\varrho(y)) \leq \varrho(x,z) + \varrho(z,y) \\
&- \varrho(z,z) - \big(T(u_\varrho(x), u_\varrho(z)) + T(u_\varrho(z), u_\varrho(y)) - u_\varrho(z)\big) \\
&= \varrho(x,z) - T(u_\varrho(x), u_\varrho(z)) + \varrho(z,y) - T(u_\varrho(z), u_\varrho(y)) \\
&= d(x,z) + d(z,y).
\end{aligned}$$

An example of a natural operator T which satisfies the requirements of Propositions 4.4, 4.5 and 4.6 is

$$T \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x,y) \mapsto \frac{1}{2}(x+y).$$

Examples of d-metrics can be found in the long version of the paper.

The following proposition demonstrates an alternative way of obtaining d-ultrametrics from ultrametrics. This result is of importance in the area of denotational semantics where ultrametric structures naturally appear.

4.7 Proposition. *Let (X, d) be an ultrametric space and let $u \colon X \to \mathbb{R}_0^+$ be a function. Then (X, ϱ) with*

$$\varrho(x,y) = \max\{d(x,y), u(x), u(y)\}$$

is a d-ultrametric and $\varrho(x,x) = u(x)$ for all $x \in X$. If u is a continuous function on (X,d), then completeness of (X,d) implies completeness of (X, ϱ).

Proof. (Mii) and (Miii) are obvious.
(Miv′) We obtain

$$\begin{aligned}
\varrho(x,y) &= \max\{d(x,y), u(x), u(y)\} \\
&\leq \max\{d(x,z), d(z,y), u(x), u(y)\} \\
&\leq \max\{d(x,z), u(x), u(z), d(z,y), u(y)\} \\
&= \max\{\varrho(x,z), \varrho(z,y)\}.
\end{aligned}$$

For completeness, let (x_n) be a Cauchy sequence in (X, ϱ). Then (x_n) is a Cauchy sequence in (X, d) and converges to some $x \in X$. We then obtain $\varrho(x_n, x) = \max\{d(x_n, x), u(x_n), u(x)\} \to u(x)$ for $n \to \infty$. As in the proof of Proposition 4.5 we obtain $u(x) = 0$ which completes the proof.

5 DISCUSSION

We have studied dislocated metric spaces and an underlying generalized notion of topology, the dislocated topology. Whilst a few applications of dislocated metrics, and in particular of the generalized Banach contraction mapping theorem, Theorem 2.2, are known in Theoretical Computer Science, it is at this stage unclear whether or not other applications can be found, and where else in Mathematics these spaces appear. If they do arise elsewhere, further theoretical investigations of dislocated metrics will be worth undertaking. The authors are currently developing further applications of Theorem 2.2 to logic programming semantics. It may also be possible to merge Theorem 2.2 and the fixed-point theorem given in [5].

Applications to logic programming hint at interpreting the dislocation function u_ϱ as a measure of *undesirability*. Whether or not this point of view is useful remains to be seen.

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